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A GENERALIZED VARIATIONAL PRINCIPLE FOR THIN ELASTIC SHELLS WITH FINITE ROTATIONS

MASASHI IURA
Department of Civil and Structural Engineering, Tokyo Denki University, Hatoyama, Hiki-gun, Saitama 350-03, Japan

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Abstract—Entirely Lagrangian nonlinear theory of thin elastic shells with finite rotations is developed. Without restriction to small strains, accurate equilibrium equations and boundary conditions are derived, utilizing the modified irrational tensor of change of curvature. The introduction of variations of displacement vectors in place of variations of displacement components makes it possible to reduce computational efforts for deriving the shell equations. With the aid of the present shell equations, the Hu-Washizu variational functional including the effects of finite rotations at the shell boundary is generated.

1. INTRODUCTION

The formulation of geometrically nonlinear theory of thin elastic shells has received much attention of research workers. Two approaches, hitherto, have been employed for such problems: one is the Eulerian formulation[1–9] and the other the Lagrangian formulation[6–18]. It is widely accepted that employing the Lagrangian formulation, rather than the Eulerian one, is desirable for numerical analysis of geometrically nonlinear shell structures.

The Lagrangian nonlinear theory of shells has been developed by many authors using the Kirchhoff–Love hypothesis. A major difficulty, encountered in the formulation of a general theory, arises from the fact that the tensor of change of curvature is an irrational function of displacements and their surface derivatives. In many papers, on the basis of small-strain assumptions, the tensor of change of curvature has been approximated in terms of polynomials of displacements and their surface derivatives. Consequently, the appropriate static and geometric boundary conditions for nonlinear shell theory have not been obtained[6–12, 17].

Pietraszkiewicz and Szwabowicz[14] have derived the nonlinear equations for the boundary conditions. However, the effects of finite rotations are not strictly taken into account, since a new tensor of change of curvature used in [14] is reduced to a third-degree polynomial. Pietraszkiewicz[13], utilizing the modified irrational tensor of change of curvature, has derived the Lagrangian equilibrium equations of shells undergoing finite rotations. However, he has failed to derive the associated boundary conditions consistent with the Lagrangian nonlinear shell theory. Pietraszkiewicz[16], as well as Iura and Hirashima[18], has employed an irrational tensor of change of curvature, defined as a difference between the curvature tensor of the deformed and undeformed shell midsurface. In case of evaluating the external virtual work for the couple, the former paper has utilized the vector, defined as a difference between the deformed and undeformed normal vector, while the latter paper has employed the total finite rotation vector. Linearizing the shell equations of both the papers, however, may not lead to the best linear theory[6].

In this paper, the fully Lagrangian nonlinear theory of thin elastic shells is developed under the Kirchhoff–Love hypothesis. Utilizing the resulting Lagrangian shell equations, the Hu–Washizu variational functional, including the effects of the finite rotations at the shell boundary, has been derived. When we derive the Lagrangian equilibrium equations and the associated boundary conditions from the principle of virtual work, we do not restrict the magnitude of rotations of shells, nor do we use the small-strain assumptions. As a tensor of change of curvature, we employ the modified irrational tensor of change of curvature introduced by Budiansky[11]. The use of this tensor, employed also in [13], may lead to the best linear theory when the results obtained here are linearized.

The appearance of finite rotations is one of the important features of any nonlinear theory of shells. The rotations have been conventionally described by a proper orthogonal tensor or a finite rotation vector. It is shown in [6–9] that the total finite rotation vector on the shell boundary is described in terms of the finite rigid-body rotation vector and another finite rotation vector. The latter rotation vector is caused by the pure stretch of the principal axes of strain. Then the external virtual work for the couple on the shell boundary has been expressed by the inner product of the total finite rotation vector and the boundary-couple vector. However, some approximations have been made on the corresponding terms in case of deriving the boundary conditions[6–9]. On the other hand, in this paper the external virtual work for the couple on the shell boundary are evaluated without approximation. As a result, the effects of finite rotations are strictly taken into account.

It appears that a cumbersome calculation is hard to be avoided in obtaining the basic shell equations. This is because we take the variations of the surface strain tensor and the tensor of change of curvature with respect to the displacement components. In this paper, in place of variations of displacement components, the variations of displacement vectors are introduced effectively. Therefore it becomes a straight forward matter to derive the equilibrium equations and the associated boundary conditions without using the small-strain assumptions. In case of the constitutive equations we assume that the shell materials consist of hyperelastic ones.

Using the basic shell equations derived here, and postulating the existence of an elastic potential function for hyperelastic materials, a generalized variational principle is generated. Starting from the principle of virtual work, the free functional, applicable to geometrically nonlinear theory of shells with unrestricted rotations, has been derived.

Throughout this paper, the summation convention will apply to repeated Greek indexes (in mixed position) with range 2.

2. NOTATIONS AND PRELIMINARIES

Let $\mathbf{r}(\theta^\alpha)$ be the position vector of the undeformed shell midsurface M, with convected curvilinear Gaussian coordinates θ^α, covariant surface base vectors $\mathbf{a}_\alpha = \mathbf{r}_{,\alpha}$ and a unit vector $\mathbf{n} = \frac{1}{2}\epsilon^{\alpha\beta}\, \mathbf{a}_\alpha \times \mathbf{a}_\beta$. The notation $(\ )_{,\alpha}$ denotes partial differentiation on M, with respect to θ^α and $\epsilon^{\alpha\beta}$ the permutation tensor of the undeformed midsurface. As usual we define covariant components of the surface metric tensor $a_{\alpha\beta} = \mathbf{a}_\alpha \cdot \mathbf{a}_\beta$ with the determinant $a = |a_{\alpha\beta}|$ and of the surface curvature tensor $b_{\alpha\beta} = \mathbf{a}_{\alpha,\beta} \cdot \mathbf{n}$. Using the Kronecker delta δ^α_β, the contravariant components of the metric tensor $a^{\alpha\beta}$ are defined by the relations $a^{\alpha\lambda}\, a_{\lambda\beta} = \delta^\alpha_\beta$.

The deformation of the shell midsurface from the undeformed reference configuration M into the deformed configuration $\overline{M}$ can be described by the displacement vector $\mathbf{u} = u^\alpha \mathbf{a}_\alpha + w\mathbf{n}$. With the deformed shell midsurface $\overline{M}$, we associate a position vector $\bar{\mathbf{r}} = \mathbf{r} + \mathbf{u}$, base vectors $\bar{\mathbf{a}}_\alpha = \bar{\mathbf{r}}_{,\alpha}$, a unit normal vector $\bar{\mathbf{n}} = \frac{1}{2}\bar{\epsilon}^{\alpha\beta}\bar{\mathbf{a}}_\alpha \times \bar{\mathbf{a}}_\beta$, surface metric tensor $\bar{a}_{\alpha\beta} = \bar{\mathbf{a}}_\alpha \cdot \bar{\mathbf{a}}_\beta$, and surface curvature tensor $\bar{b}_{\alpha\beta} = \bar{\mathbf{a}}_{\alpha,\beta} \cdot \bar{\mathbf{n}}$. For the base vectors the following relations are satisfied[1]:

$$\bar{\mathbf{a}}_\alpha = l_{\lambda\alpha}\mathbf{a}^\lambda + \phi_\alpha \mathbf{n} = \mathbf{a}_\alpha + \mathbf{u}_{,\alpha}, \tag{1a}$$

$$\bar{\mathbf{n}} = n_\alpha \mathbf{a}^\alpha + n\mathbf{n}, \tag{1b}$$

$$l_{\alpha\beta} = a_{\alpha\beta} + \theta_{\alpha\beta} - \omega_{\alpha\beta}, \tag{1c}$$

$$\theta_{\alpha\beta} = \tfrac{1}{2}(u_\alpha|_\beta + u_\beta|_\alpha) - b_{\alpha\beta}w, \tag{1d}$$

$$\omega_{\alpha\beta} = \tfrac{1}{2}(u_\beta|_\alpha - u_\alpha|_\beta), \tag{1e}$$

$$\phi_\alpha = w_{,\alpha} + b^\lambda_\alpha u_\lambda, \tag{1f}$$

$$n_\mu = \sqrt{\left(\frac{a}{\bar{a}}\right)}\, \epsilon^{\alpha\beta}\, \epsilon_{\lambda\mu}\, \phi_\alpha\, l^{\lambda}{}_{\cdot\beta}, \tag{1g}$$

$$n = \frac{1}{2}\sqrt{\left(\frac{a}{\bar{a}}\right)}\, \epsilon^{\alpha\beta}\, \epsilon_{\lambda\mu}\, l^{\lambda}{}_{\cdot\alpha}\, l^{\mu}{}_{\cdot\beta}, \tag{1h}$$

where $(\ )|_\alpha$ denotes the surface covariant differentiation at M. The displacement field at an arbitrary point of the shell with the distance ζ from M is represented by

$$\mathbf{v} = \mathbf{u} + \zeta\, \boldsymbol{\beta}, \tag{2}$$

where

$$\boldsymbol{\beta} = \bar{\mathbf{n}} - \mathbf{n}. \tag{3}$$

Assuming that the Kirchhoff–Love hypothesis holds, the shell deformation can be described by the surface strain tensor $\gamma_{\alpha\beta}$ and the tensor of change of curvature $\kappa_{\alpha\beta}$ defined by

$$\gamma_{\alpha\beta} = \tfrac{1}{2}(\bar{a}_{\alpha\beta} - a_{\alpha\beta}), \tag{4a}$$

$$\kappa_{\alpha\beta} = -(\bar{b}_{\alpha\beta} - b_{\alpha\beta}). \tag{4b}$$

In this paper, instead of $\kappa_{\alpha\beta}$, we employ the following modified tensor of change of curvature $\rho_{\alpha\beta}$ introduced by Budiansky[11]:

$$\rho_{\alpha\beta} = \kappa_{\alpha\beta} + \tfrac{1}{2}(b^{\kappa}_{\alpha}\gamma_{\kappa\beta} + b^{\kappa}_{\beta}\gamma_{\kappa\alpha}). \tag{5}$$

It should be noted that the present tensor $\rho_{\alpha\beta}$ has the attraction in discussing on general theorems in the linear theory, and that it is considered the best choice from the point of view of a number of criteria[1, 11, 13]. Note also that the tensor $\rho_{\alpha\beta}$ is, in general, an irrational function of the displacements, since it contains an invariant expressed by

$$\frac{\bar{a}}{a} = 1 + 2\,\gamma^{\alpha}_{\alpha} + 2\,(\gamma^{\alpha}_{\alpha}\gamma^{\beta}_{\beta} - \gamma^{\alpha}_{\beta}\gamma^{\beta}_{\alpha}). \tag{6}$$

In the existing literature, because of complex form of $\kappa_{\alpha\beta}$ or $\rho_{\alpha\beta}$, various variants of approximate strain–displacement relations have been used on the basis of small-strain assumptions. This is why there exist few investigations in which the effects of finite rotations are strictly taken into account.

Substituting eqns (1) into eqns (4) and (5) yields the strain–displacement relations. The variation of the modified tensor of change of curvature in terms of displacement components takes complex forms, since it contains the irrational function $\sqrt{(a/\bar{a})}$. Consequently, extensive computations may be needed to derive the fundamental equations of the shell. In this work, the variation of displacement vectors in place of displacement components is used effectively. The variation of displacement vectors has been used also by Pietraszkiewicz[13], though appropriate boundary conditions have not been obtained. This is owing to the fact that the variation of derivatives of displacement vectors in the outward normal direction could not have been eliminated through integration by parts along the boundary. The advantages of the usage of such variations are that the variations of $\gamma_{\alpha\beta}$ and $\rho_{\alpha\beta}$ take more simple forms, and that computational efforts for deriving the shell equations are significantly reduced. With the aid of the orthogonality of $\bar{\mathbf{a}}_\alpha$ and $\bar{\mathbf{n}}$, we obtain the variation of the surface strain tensor and the modified tensor of change of curvature in terms of displacement vectors, represented by

$$\delta\gamma_{\alpha\beta} = \tfrac{1}{2}\,(\delta\mathbf{u}_{,\alpha}\cdot\bar{\mathbf{a}}_\beta + \bar{\mathbf{a}}_\alpha\cdot\delta\mathbf{u}_{,\beta}), \tag{7a}$$

$$\delta\rho_{\alpha\beta} = -[(\delta\mathbf{u}_{,\alpha})\,|_\beta - \bar{a}^{\kappa\lambda}\gamma_{\lambda\alpha\beta}\delta\mathbf{u}_{,\kappa}]\cdot\bar{\mathbf{n}} + \tfrac{1}{2}\,(b^\kappa_\alpha\delta\gamma_{\kappa\beta} + b^\kappa_\beta\,\delta\gamma_{\kappa\beta}), \tag{7b}$$

where

$$\gamma_{\lambda\alpha\beta} = \gamma_{\lambda\alpha}\,|\,_\beta + \gamma_{\lambda\beta}\,|\,_\alpha - \gamma_{\alpha\beta}\,|\,_\lambda. \tag{8}$$

It should be emphasized that the variations of $\gamma_{\alpha\beta}$ and $\rho_{\alpha\beta}$ are linear functions of $\delta\mathbf{u}$.

3. DEFORMATION OF BOUNDARY ELEMENTS

Let C be the boundary contour at M, defined by the equation $\theta^\alpha = \theta^\alpha(s)$, where s is the length parameter of C. We assume that in the reference configuration the lateral shell boundary surface is rectilinear and orthogonal to M along C. With the boundary contour C, we associate the unit tangent vector $\mathbf{t} = \mathbf{r}_{,s}$, where $(\ )_{,s}$ denotes the differentiation with respect to s, and the outward unit normal vector $\boldsymbol{\nu} = \mathbf{t} \times \mathbf{n}$.

After the shell deformation under the Kirchhoff–Love hypothesis, the orthonormal triad $\boldsymbol{\nu}$, $\mathbf{t}$ and $\mathbf{n}$ is transformed into an orthogonal triad $\bar{\mathbf{a}}_\nu$, $\bar{\mathbf{a}}_t$ and $\bar{\mathbf{n}}$, defined by

$$\bar{\mathbf{a}}_t = \bar{\mathbf{r}}_{,s} = t^\alpha\bar{\mathbf{a}}_\alpha, \tag{9a}$$

$$\bar{\mathbf{a}}_\nu = \bar{\mathbf{a}}_t \times \bar{\mathbf{n}} = \sqrt{\left(\frac{\bar{a}}{a}\right)}\,\nu_\alpha\bar{\mathbf{a}}^\alpha, \tag{9b}$$

$$|\,\bar{\mathbf{a}}_t\,| = |\,\bar{\mathbf{a}}_\nu\,| = \bar{a}_t = \sqrt{(1 + 2\,\gamma_{tt})}, \tag{9c}$$

$$\gamma_{tt} = \gamma_{\alpha\beta}\,t^\alpha\,t^\beta. \tag{9d}$$

According to the polar decomposition theorem, the boundary deformation can be decomposed into a rigid-body translation, a pure strain along principal axes of strain and a rigid-body rotation of the principal axes. Since the axes defined by $\boldsymbol{\nu}$ and $\mathbf{t}$ do not coincide, in general, with the principal axes of strain, the vectors $\boldsymbol{\nu}$ and $\mathbf{t}$ not only change their lengths, but also yield rotations during the pure stretch of the principal axes of strain. Accordingly, the total rotation vector $\boldsymbol{\Omega}_t$ of the orthonormal vectors $\boldsymbol{\nu}$, $\mathbf{t}$ and $\mathbf{n}$ is composed of the finite rigid-body rotation vector $\boldsymbol{\Omega}$, and the finite rotation vector $\hat{\boldsymbol{\Omega}}$ of the boundary caused by the pure stretch of the principal axes of strain. The superposition rules for the finite rotation vectors are different from the usual addition rules for a linear vector space. The relationships between these rotation vectors are detailed in [7]. In the Lagrangian description, the transformation of $\boldsymbol{\nu}$, $\mathbf{t}$ and $\mathbf{n}$ into $\bar{\mathbf{a}}_\nu$, $\bar{\mathbf{a}}_t$ and $\bar{\mathbf{n}}$ consists of extension by the factor $\bar{a}_t$, which causes no extension in $\mathbf{n}$, and the two successive rotations: first through $\hat{\boldsymbol{\Omega}}_t$, then through $\boldsymbol{\Omega}$. The transformation of the vectors $\boldsymbol{\nu}$, $\mathbf{t}$ and $\mathbf{n}$ into the vectors $\bar{\mathbf{a}}_\nu$, $\bar{\mathbf{a}}_t$ and $\bar{\mathbf{n}}$ becomes

$$\bar{\mathbf{a}}_\nu = \bar{a}_t\left[\boldsymbol{\nu} + \boldsymbol{\Omega}_t \times \boldsymbol{\nu} + \frac{\boldsymbol{\Omega}_t \times (\boldsymbol{\Omega}_t \times \boldsymbol{\nu})}{2\cos^2\omega_t/2}\right], \tag{10a}$$

$$\bar{\mathbf{a}}_t = \bar{a}_t\left[\mathbf{t} + \boldsymbol{\Omega}_t \times \mathbf{t} + \frac{\boldsymbol{\Omega}_t \times (\boldsymbol{\Omega}_t \times \mathbf{t})}{2\cos^2\omega_t/2}\right], \tag{10b}$$

$$\bar{\mathbf{n}} = \left[\mathbf{n} + \boldsymbol{\Omega}_t \times \mathbf{n} + \frac{\boldsymbol{\Omega}_t \times (\boldsymbol{\Omega}_t \times \mathbf{n})}{2\cos^2\omega_t/2}\right], \tag{10c}$$

where

$$\sin\omega_t = |\,\boldsymbol{\Omega}_t\,|. \tag{11}$$

From eqns (10) we have the following relations:

$$2\,\mathbf{\Omega}_{,t}\cdot\boldsymbol{\nu} = \bar{\mathbf{t}}\cdot\mathbf{n} - \bar{\mathbf{n}}\cdot\mathbf{t}, \tag{12a}$$

$$2\,\mathbf{\Omega}_{,t}\cdot\mathbf{t} = \bar{\mathbf{n}}\cdot\boldsymbol{\nu} - \bar{\boldsymbol{\nu}}\cdot\mathbf{n}, \tag{12b}$$

$$2\,\mathbf{\Omega}_{,t}\cdot\mathbf{n} = \bar{\boldsymbol{\nu}}\cdot\mathbf{t} - \bar{\mathbf{t}}\cdot\boldsymbol{\nu}, \tag{12c}$$

where the unit vectors $\bar{\boldsymbol{\nu}}$ and $\bar{\mathbf{t}}$ are defined as

$$\bar{\boldsymbol{\nu}} = \bar{\mathbf{a}}_\nu/\bar{a}_t, \tag{13a}$$

$$\bar{\mathbf{t}} = \bar{\mathbf{a}}_t/\bar{a}_t. \tag{13b}$$

Equations (12) will be used in evaluating the external virtual work on the boundary. For the latter convenience, we decompose the vector $\boldsymbol{\beta}$, defined by eqn (3), with respect to the orthonormal triad $\boldsymbol{\nu}$, $\mathbf{t}$ and $\mathbf{n}$ in the form $\boldsymbol{\beta} = \beta_\nu\boldsymbol{\nu} + \beta_t\mathbf{t} + \beta\mathbf{n}$.

4. INTERNAL VIRTUAL WORK

Let the shell be equilibrium under the external surface forces and boundary forces and couples, the directions of which are assumed to remain constant during deformation. The Lagrangian equilibrium equations and the associated static and geometric boundary conditions for shells may be derived from the principle of virtual work. For any additional virtual displacement vector $\delta\mathbf{u} = \delta u^\alpha\mathbf{a}_\alpha + \delta w\mathbf{n}$, subjected to geometric constraints, there should be Lagrangian symmetric (2nd Piola–Kirchhoff type) stress resultant tensors $N^{\alpha\beta}$ and couple resultant tensors $M^{\alpha\beta}$, such that the Lagrangian internal virtual work takes the form

$$\mathrm{IVW} = \iint_M (N^{\alpha\beta}\delta\gamma_{\alpha\beta} + M^{\alpha\beta}\delta\rho_{\alpha\beta})\,\mathrm{d}A. \tag{14}$$

Substituting eqns (7) into eqn (14), and using the Stokes theorem, yield

$$\mathrm{IVW} = -\iint_M \mathbf{T}^\beta|_\beta\cdot\delta\mathbf{u}\,\mathrm{d}A + \int_C \mathbf{T}^\beta\,\nu_\beta\cdot\delta\mathbf{u}\,\mathrm{d}s - \int_C M^{\alpha\beta}\,t_\beta\,\nu_\alpha\bar{\mathbf{n}}\cdot\delta\mathbf{u}_{,s}\,\mathrm{d}s - \int_C M^{\alpha\beta}\,\nu_\alpha\,\nu_\beta\bar{\mathbf{n}}\cdot\delta\mathbf{u}_{,\nu}\,\mathrm{d}s, \tag{15}$$

where

$$\mathbf{T}^\beta = T^{\alpha\beta}\,\bar{\mathbf{a}}_\alpha + Q^\beta\bar{\mathbf{n}}, \tag{16a}$$

$$T^{\alpha\beta} = N^{\alpha\beta} - \bar{b}^\alpha_\kappa\,M^{\kappa\beta} + \tfrac{1}{2}\,(b^\alpha_\kappa\,M^{\kappa\beta} + b^\beta_\kappa\,M^{\alpha\kappa}), \tag{16b}$$

$$Q^\beta = M^{\alpha\kappa}\,\bar{a}^{\beta\lambda}\,\gamma_{\lambda\alpha\kappa} + M^{\alpha\beta}|_\alpha. \tag{16c}$$

In general, under the Kirchhoff–Love hypothesis, the internal virtual work on the boundary should be expressed in terms of the variations of the four independent parameters. In the existing literature[6–9, 13], an approximation has been made on the last term on the right-hand side of eqn (15), since the inner product $\bar{\mathbf{n}}\cdot\delta\mathbf{u}_{,\nu}$ could not have been expressed in terms of the variations of the four independent parameters. As a result, accurate boundary conditions for the couple could not have been obtained. In this paper it will be shown after some transformation that the internal and external virtual works on the boundary can be expressed in terms of the variations of the displacement vector and the fourth parameter.

Let us consider the inner product $\bar{\mathbf{n}}\cdot\delta\mathbf{u}_{,\nu}$. Using the definitions

$$\bar{\mathbf{n}} = \tfrac{1}{2}\,\bar{\epsilon}^{\alpha\beta}\bar{\mathbf{a}}_\alpha \times \bar{\mathbf{a}}_\beta, \qquad \bar{\epsilon}^{\alpha\beta} = \sqrt{\left(\frac{a}{\bar{a}}\right)}\,\epsilon^{\alpha\beta},$$

and observing that

$$\bar{\mathbf{a}}_\alpha = \nu_\alpha\,\bar{\mathbf{r}}_{,\nu} + t_\alpha\bar{\mathbf{r}}_{,s}, \tag{17}$$

we can write the normal vector after deformation in the form

$$\bar{\mathbf{n}} = \sqrt{\left(\frac{a}{\bar{a}}\right)}\,\bar{\mathbf{r}}_{,\nu} \times \bar{\mathbf{r}}_{,s}. \tag{18}$$

Using the relation $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$, and substituting eqn (18) into eqn (9b), leads to

$$\bar{\mathbf{a}}_\nu = \sqrt{\left(\frac{a}{\bar{a}}\right)}\,(\bar{a}_t^2\bar{\mathbf{r}}_{,\nu} - \bar{b}_t\bar{\mathbf{r}}_{,s}), \tag{19}$$

where

$$\bar{b}_t = \nu^\alpha t^\beta \bar{a}_{\alpha\beta}. \tag{20}$$

From eqn (19), we have

$$\bar{\mathbf{r}}_{,\nu} = \frac{\bar{b}_t}{\bar{a}_t^2}\,\bar{\mathbf{a}}_t + \frac{1}{\bar{a}_t^2}\sqrt{\left(\frac{\bar{a}}{a}\right)}\,\bar{\mathbf{a}}_\nu. \tag{21}$$

With the help of eqn (21) and the condition of orthogonality of the vectors $\bar{\mathbf{a}}_\nu$, $\bar{\mathbf{a}}_t$ and $\bar{\mathbf{n}}$, the inner product $\bar{\mathbf{n}}\cdot\delta\mathbf{u}_{,\nu}$ is presented by

$$\bar{\mathbf{n}}\cdot\delta\mathbf{u}_{,\nu} = \frac{\bar{b}_t}{\bar{a}_t^2}\,\bar{\mathbf{n}}\cdot\delta\mathbf{u}_{,s} - \frac{1}{\bar{a}_t^2}\sqrt{\left(\frac{\bar{a}}{a}\right)}\,\bar{\mathbf{a}}_\nu\cdot\delta\bar{\mathbf{n}}. \tag{22}$$

Denoting the displacement vector with respect to the orthonormal triad

$$\mathbf{u} = u_\nu\boldsymbol{\nu} + u_t\mathbf{t} + w\,\mathbf{n}, \tag{23}$$

the tangent vector of the deformed boundary is written by

$$\bar{\mathbf{a}}_t = c_\nu\boldsymbol{\nu} + c_t\mathbf{t} + c\,\mathbf{n}, \tag{24}$$

where

$$c_\nu = u_{\nu,s} + \tau_t\,w - \kappa_t\,u_t, \tag{25a}$$

$$c_t = 1 + u_{t,s} + \kappa_t\,u_\nu - \sigma_t w, \tag{25b}$$

$$c = w_{,s} + \sigma_t\,u_t - \tau_t\,u_\nu, \tag{25c}$$

and σ_t denotes a normal curvature, τ_t a geodesic torsion and κ_t a geodesic curvature of the surface boundary contour C. From the definition $\bar{\mathbf{a}}_\nu = \bar{\mathbf{a}}_t \times \bar{\mathbf{n}}$ and eqn (24), we may write the inner product $\bar{\mathbf{a}}_\nu\cdot\delta\bar{\mathbf{n}}$ in the form

$$\bar{\mathbf{a}}_\nu\cdot\delta\bar{\mathbf{n}} = \mathrm{d}_\nu\,\delta\beta_\nu + \mathrm{d}_t\,\delta\beta_t + \mathrm{d}\,\delta\beta, \tag{26}$$

where

$$d_\nu = c_t\,(1 + \beta) - c\,\beta_t, \tag{27a}$$

$$d_t = c\,\beta_\nu - c_\nu\,(1 + \beta), \tag{27b}$$

$$d = c_\nu\,\beta_t - c_t\,\beta_\nu. \tag{27c}$$

It should be stressed that, under the Kirchhoff–Love hypothesis, the variations $\delta\beta_t$ and $\delta\beta$ are dependent variables of the variations $\delta\mathbf{u}_{,s}$ and $\delta\beta_\nu$. Since

$$\boldsymbol{\beta}\cdot\boldsymbol{\beta} = -\,2\,\mathbf{n}\cdot\boldsymbol{\beta}, \tag{28}$$

we have

$$(\beta_\nu)^2 + (\beta_t)^2 + (\beta)^2 = -\,2\,\beta. \tag{29}$$

Forming the variation of eqn (29), we obtain

$$\delta\beta = -\,\frac{\beta_\nu\,\delta\beta_\nu + \beta_t\,\delta\beta_t}{1 + \beta}. \tag{30}$$

On the other hand, it follows from $\delta(\bar{\mathbf{a}}_t\cdot\bar{\mathbf{n}}) = 0$ that

$$\delta\mathbf{u}_{,s}\cdot\bar{\mathbf{n}} + c_\nu\,\delta\beta_\nu + c_t\,\delta\beta_t + c\,\delta\beta = 0. \tag{31}$$

Introducing eqn (30) into eqn (31), the variation $\delta\beta_t$ can be expressed in the form

$$\delta\beta_t = f_t\,\bar{\mathbf{n}}\cdot\delta\mathbf{u}_{,s} + f_\nu\,\delta\beta_\nu, \tag{32}$$

where

$$f_t = -\,\frac{1 + \beta}{c_t\,(1 + \beta) - c\,\beta_t}, \tag{33a}$$

$$f_\nu = -\,\frac{c_\nu\,(1 + \beta) - c\,\beta_\nu}{c_t\,(1 + \beta) - c\,\beta_t}. \tag{33b}$$

Substituting eqn (32) into eqn (30), the variation $\delta\beta$ can be written in the form

$$\delta\beta = g_t\bar{\mathbf{n}}\cdot\delta\mathbf{u}_{,s} + g_\nu\,\delta\beta_\nu, \tag{34}$$

where

$$g_t = -\,\frac{f_t\beta_t}{1 + \beta}, \tag{35a}$$

$$g_\nu = -\,\frac{\beta_\nu + f_\nu\beta_t}{1 + \beta}. \tag{35b}$$

From eqns (22), (26), (32) and (34), we obtain the inner product $\bar{\mathbf{n}}\cdot\delta\mathbf{u}_{,\nu}$, expressed in terms of the variations $\delta\mathbf{u}_{,s}$ and $\delta\beta_\nu$ as

$$\bar{\mathbf{n}}\cdot\delta\mathbf{u}_{,\nu} = -\,h_t\bar{\mathbf{n}}\cdot\delta\mathbf{u}_{,s} - h_\nu\,\delta\beta_\nu, \tag{36}$$

where

$$h_t = -\frac{\bar{b}_t}{\bar{a}_t^2} + \frac{1}{\bar{a}_t^2}\sqrt{\left(\frac{\bar{a}}{a}\right)}\,(d_t\, f_t + d\, g_t), \tag{37a}$$

$$h_\nu = \frac{1}{\bar{a}_t^2}\sqrt{\left(\frac{\bar{a}}{a}\right)}\,(d_\nu + d_t\, f_\nu + d\, g_\nu). \tag{37b}$$

Using eqn (36) the internal virtual work can be rewritten in the form

$$\mathrm{IVW} = -\iint_M \mathbf{T}^\beta|_\beta \cdot \delta\mathbf{u}\, dA + \int_C (\mathbf{T}\cdot\delta\mathbf{u} + M_{\nu\nu}\delta\beta_\nu)\, dS + \sum_k \mathbf{R}_k\cdot\delta\mathbf{u}_k, \tag{38}$$

where

$$\mathbf{T} = \mathbf{T}^\beta\, \nu_\beta + \mathbf{R}_{,s}, \tag{39a}$$

$$\mathbf{R} = R\,\bar{\mathbf{n}}, \tag{39b}$$

$$R = M^{\alpha\beta}\,\nu_\alpha\,(t_\beta - \nu_\beta\, h_t), \tag{39c}$$

$$M_{\nu\nu} = M^{\alpha\beta}\,\nu_\alpha\,\nu_\beta\, h_\nu, \tag{39d}$$

$$\mathbf{R}_k = \mathbf{R}(S_k + 0) - \mathbf{R}(S_k - 0), \tag{39e}$$

and M_k, $k = 1, 2, \ldots, K$, are corner points of the boundary contour, and S_k denote the coordinates corresponding to the corner points.

It should be emphasized that the internal virtual work on the bounary can be expressed in terms of the variations of the displacement vector $\mathbf{u}$ and the fourth parameter β_ν describing the finite rotation of the shell boundary.

5. EXTERNAL VIRTUAL WORK

Let us consider a shell subjected to the surface force $\mathbf{P} = p^\alpha \mathbf{a}_\alpha + p\mathbf{n}$ per unit area of the undeformed middle surface, the boundary force $\mathbf{F} = F_\nu \boldsymbol{\nu} + F_t \mathbf{t} + F\,\mathbf{n}$, and the bondary couple $\mathbf{K} = -k_t \boldsymbol{\nu} + k_\nu \mathbf{t} + k\,\mathbf{n}$ per unit length of the undeformed boundary. Then the Lagrangian external virtual work can be put in the form[6–9]

$$\mathrm{EVW} = \iint_M \mathbf{P}\cdot\delta\mathbf{u}\, dA + \int_C (\mathbf{F}\cdot\delta\mathbf{u} + \mathbf{K}\cdot\delta\boldsymbol{\Omega}_t)\, dS. \tag{40}$$

The inner product $\mathbf{K}\cdot\delta\boldsymbol{\Omega}_t$ must be expressed in terms of the variations of the four independent parameters. With the help of eqns (12), the inner products of the orthonormal triad $\boldsymbol{\nu}$, $\mathbf{t}$, $\mathbf{n}$ and the variation of the total finite rotation vector are expressed by

$$\boldsymbol{\nu}\cdot\delta\boldsymbol{\Omega}_t = \tfrac{1}{2}\,\delta(\bar{\mathbf{t}}\cdot\mathbf{n} - \bar{\mathbf{n}}\cdot\mathbf{t}), \tag{41a}$$

$$\mathbf{t}\cdot\delta\boldsymbol{\Omega}_t = \tfrac{1}{2}\,\delta(\bar{\mathbf{n}}\cdot\boldsymbol{\nu} - \bar{\boldsymbol{\nu}}\cdot\mathbf{n}), \tag{41b}$$

$$\mathbf{n}\cdot\delta\boldsymbol{\Omega} = \tfrac{1}{2}\,\delta(\bar{\boldsymbol{\nu}}\cdot\mathbf{t} - \bar{\mathbf{t}}\cdot\boldsymbol{\nu}). \tag{41c}$$

Substituting eqns (32) and (34) into eqns (41) leads to

$$\boldsymbol{\nu}\cdot\delta\boldsymbol{\Omega}_t = \mathbf{Q}_\nu\cdot\delta\mathbf{u}_{,s} + Q_\nu\,\delta\beta_\nu, \tag{42a}$$

$$\mathbf{t}\cdot\delta\boldsymbol{\Omega}_t = \mathbf{Q}_t\cdot\delta\mathbf{u}_{,s} + Q_t\,\delta\beta_\nu, \tag{42b}$$

$$\mathbf{n}\cdot\delta\boldsymbol{\Omega}_t = \mathbf{Q}_n\cdot\delta\mathbf{u}_{,s} + Q_n\,\delta\beta_\nu, \tag{42c}$$

where

$$\mathbf{Q}_\nu = q_{\nu\nu}\,\boldsymbol{\nu} + q_{\nu t}\,\mathbf{t} + q_{\nu n}\,\mathbf{n}, \tag{43a}$$

$$\mathbf{Q}_t = q_{t\nu}\,\boldsymbol{\nu} + q_{tt}\,\mathbf{t} + q_{tn}\,\mathbf{n}, \tag{43b}$$

$$\mathbf{Q}_n = q_{n\nu}\,\boldsymbol{\nu} + q_{nt}\,\mathbf{t} + q_{nn}\,\mathbf{n}, \tag{43c}$$

$$Q_\nu = -\tfrac{1}{2} f_\nu, \tag{43d}$$

$$Q_t = \frac{1}{2\,\overline{a}_t}(\overline{a}_t - c_\nu f_\nu + c_t), \tag{43e}$$

$$Q = \frac{1}{2\,\overline{a}_t}(c - c_\nu g_\nu), \tag{43f}$$

$$q_{\nu\nu} = \frac{1}{2}\left(-\frac{c\,c_\nu}{\overline{a}_t^3} - f_t\,n^\alpha\,\nu_\alpha\right), \tag{43g}$$

$$q_{\nu t} = \frac{1}{2}\left(-\frac{c\,c_t}{\overline{a}_t^3} - f_t\,n^\alpha\,t_\alpha\right), \tag{43h}$$

$$q_{\nu n} = \frac{1}{2}\left(\frac{1}{\overline{a}_t} - \frac{c^2}{\overline{a}_t^3} - f_n\,n\right), \tag{43i}$$

$$q_{t\nu} = \frac{1}{2}\left(-\frac{c_\nu f_t n^\alpha \nu_\alpha}{\overline{a}_t} - \frac{\beta_t}{\overline{a}_t} + \frac{c_\nu\,(c_\nu\beta_t - c_t\beta_\nu)}{\overline{a}_t^3}\right), \tag{43j}$$

$$q_{tt} = \frac{1}{2}\left(-\frac{c_\nu f_t n^\alpha t_\alpha}{\overline{a}_t} + \frac{\beta_\nu}{\overline{a}_t} + \frac{c_t\,(c_\nu\beta_t - c_t\beta_\nu)}{\overline{a}_t^3}\right), \tag{43k}$$

$$q_{tn} = \frac{1}{2}\left(-\frac{c_\nu f_t n}{\overline{a}_t} + \frac{c\,(c_\nu\beta_t - c_t\beta_\nu)}{\overline{a}_t^3}\right), \tag{43l}$$

$$q_{n\nu} = \frac{1}{2}\left(-\frac{2+\beta}{\overline{a}_t} - \frac{c_\nu[c\beta_\nu - c_\nu(2+\beta)]}{\overline{a}_t^3} - \frac{c_\nu g_t n^\alpha \nu_\alpha}{\overline{a}_t}\right), \tag{43m}$$

$$q_{nt} = \frac{1}{2}\left(-\frac{c_t[c\beta_\nu - c_\nu(2+\beta)]}{\overline{a}_t^3} - \frac{c_\nu g_t n^\alpha t_\alpha}{\overline{a}_t}\right), \tag{43n}$$

$$q_{nn} = \frac{1}{2}\left(\frac{\beta_\nu}{\overline{a}_t} - \frac{c\,[c\beta_\nu - c_\nu(2+\beta)]}{\overline{a}_t^3} - \frac{c_\nu q_t n}{\overline{a}_t}\right). \tag{43o}$$

Introducing eqns (42) into eqn (40) and integrating by parts, the external virtual work can be written in terms of the variations $\delta\mathbf{u}$ and $\delta\beta_\nu$ as

$$\mathrm{EVW} = \iint_M \mathbf{P}\cdot\delta\mathbf{u}\,\mathrm{d}A + \int_{C_1}(\mathbf{T}^*\cdot\delta\mathbf{u} + M^*_{\nu\nu}\delta\beta_\nu)\,\mathrm{d}S + \sum_j \mathbf{R}^*_j\cdot\delta\mathbf{u}_j, \tag{44}$$

where

$$\mathbf{T}^* = \mathbf{F} + \mathbf{R}^*{}_{,s}, \tag{45a}$$

$$M^*_{\nu\nu} = k_\nu\,Q_t - k_t\,Q_\nu + k\,Q_n, \tag{45b}$$

$$\mathbf{R}^* = -k_\nu\,\mathbf{Q}_t + k_t\,\mathbf{Q}_\nu - k\,\mathbf{Q}_n, \tag{45c}$$

$$\mathbf{R}^*_j = \mathbf{R}^*(S_j + 0) - \mathbf{R}^*(S_j - 0), \tag{45d}$$

and C_1 is the part of C on which at least one components of $\mathbf{T}^*$ or $M^*_{\nu\nu}$ is prescribed, while $M_j, j = 1, 2, \ldots, J$, are those corner points of C where at least one component of $\mathbf{R}_j^*$ is prescribed.

6. BASIC SHELL EQUATIONS

The principle of virtual work states that the internal virtual work IVW should be equal to the external virtual work EVW. Hence, by the use of eqns (38) and (44), we obtain the Lagrangian equilibrium equations, represented by

$$\mathbf{T}^{\beta}|_{\beta} + \mathbf{P} = \mathbf{0} \qquad \text{in } M, \tag{46}$$

and the associated static boundary conditions

$$\mathbf{T} = \mathbf{T}^* \qquad \text{and} \qquad M_{\nu\nu} = M_{\nu\nu}^* \qquad \text{on} \quad C_1, \tag{47a}$$

$$\mathbf{R}_j = \mathbf{R}_j^* \qquad \text{at each} \quad M_j \in C_1. \tag{47b}$$

The geometric boundary conditions take the form

$$\mathbf{u} = \mathbf{u}^* \qquad \text{and} \qquad \beta_\nu = \beta_\nu^* \qquad \text{on} \quad C_2, \tag{48a}$$

$$\mathbf{u}_i = \mathbf{u}_i^* \qquad \text{at each} \quad M_i \in C_2, \tag{48b}$$

where C_2 is the part of C on which at least one component of $\mathbf{u}^*$ or β_ν^* is prescribed, while M_i, $i = 1, 2, \ldots, I \leqq K$, are those corner points of C_2 where at least one component of $\mathbf{u}_i^*$ is prescribed.

Assuming that a shell consists of hyperelastic materials, an elastic potential function Σ, per unit area of M, exists, such that

$$N^{\alpha\beta} = \frac{1}{2}\left(\frac{\partial \Sigma}{\partial \gamma_{\alpha\beta}} + \frac{\partial \Sigma}{\partial \gamma_{\beta\alpha}}\right), \tag{49a}$$

$$M^{\alpha\beta} = \frac{1}{2}\left(\frac{\partial \Sigma}{\partial \rho_{\alpha\beta}} + \frac{\partial \Sigma}{\partial \rho_{\beta\alpha}}\right). \tag{49b}$$

Within the consistent first-approximation theory[1, 6–9], in which strains of the shell are assumed to be small, the elastic potential function Σ^* is given by

$$\Sigma^* = \frac{1}{2} H^{\alpha\beta\lambda\mu}\left(\gamma_{\alpha\beta}\,\gamma_{\lambda\mu} + \frac{h^2}{12}\rho_{\alpha\beta}\,\rho_{\lambda\mu}\right), \tag{50}$$

where

$$H^{\alpha\beta\lambda\mu} = \frac{E}{2(1+\nu)}\left(a^{\alpha\lambda}a^{\beta\mu} + a^{\alpha\mu}a^{\beta\lambda} + \frac{2\nu}{1-\nu}a^{\alpha\beta}a^{\lambda\mu}\right), \tag{51}$$

and h denotes the thickness of the shell, E the Young modulus, and ν the Poisson ratio.

Expressing the vector equilibrium equations (46) by the component form in the bases $\mathbf{a}_\alpha$ and $\mathbf{n}$, they become

$$(l^{\alpha}{}_{.\kappa}T^{\kappa\beta} + n^{\alpha}Q^{\beta})|_{\beta} - b^{\alpha}_{\beta}(\phi_\kappa T^{\kappa\beta} + n\,Q^{\beta}) + p^{\alpha} = 0, \tag{52a}$$

$$b_{\alpha\beta}\,(l^{\alpha}{}_{.\kappa}T^{\kappa\beta} + n^{\alpha}Q^{\beta}) + (\phi_\kappa T^{\kappa\beta} + n\,Q^{\beta})|_{\beta} + p = 0. \tag{52b}$$

In a similar way, the static boundary conditions of forces in the component form, with respect to the reference triad of the vectors $\boldsymbol{\nu}$, $\mathbf{t}$ and $\mathbf{n}$, are written in the form

$$(T^{\alpha\beta}l^{\kappa}{}_{.\alpha}\nu_\beta + Q^{\beta}n^{\kappa}\nu_\beta + R_{,s}n^{\kappa} - R\,\bar{b}^{\eta}_{\mu}t^{\mu}l^{\kappa}{}_{.\eta})\nu_\kappa = F_\nu + \bar{I}_{1,s} - \kappa_t\bar{I}_2 + \tau_t\bar{I}_3, \tag{53a}$$

$$(T^{\alpha\beta}l^{\kappa}{}_{.\alpha}\nu_{\beta} + Q^{\beta}n^{\kappa}\nu_{\beta} + R_{,s}n^{\kappa} - R\bar{b}^{\eta}_{\mu}t^{\mu}l^{\kappa}{}_{.\eta})t_{\kappa} = F_t + \bar{l}_{2,s} + \kappa_t\bar{l}_1 - \sigma_t\bar{l}_3, \tag{53b}$$

$$T^{\alpha\beta}\phi_{\alpha}\nu_{\beta} + Q^{\beta}n\nu_{\beta} + R_{,s}n - R\bar{b}^{\eta}_{\mu}t^{\mu}\phi_{\eta} = F + \bar{l}_{3,s} - \tau_t\bar{l}_1 + \sigma_t\bar{l}_2, \tag{53c}$$

where

$$\bar{l}_1 = -k_\nu q_{t\nu} + k_t q_{\nu\nu} - k\,q_\nu, \tag{54a}$$

$$\bar{l}_2 = -k_\nu q_{tt} + k_t q_{\nu t} - k\,q_t, \tag{54b}$$

$$\bar{l}_3 = -k_\nu q_{tn} + k_t q_{\nu n} - k\,q_n. \tag{54c}$$

7. GENERALIZED VARIATIONAL PRINCIPLES

Variational principles have played an important role in a numerical analysis of shell structures. In the nonlinear shell theory[19–22], several variational functionals have been constructed. Most of the existing functionals, however, can be available only to shells with small or moderate rotations. Recently the generalized variational principles have been derived[20, 21] on the basis of the nonlinear theory obtained by Pietraszkiewicz and Szwabowicz[14]. As discussed later, the use of a new tensor of change of curvature employed in [14] indicates that the small-strain assumptions are introduced from the outset. As a result, the range of the application of the existing generalized variational principle is limited to shells with small strains.

In this paper the generalized variational principle is derived for the geometrically nonlinear theory of thin elastic shells with finite rotations and finite strains. For hyperelastic materials, the internal virtual work can be expressed as a variation of the elastic potential function: $\delta\,\Sigma\,(\gamma_{\alpha\beta}, \rho_{\alpha\beta}) = N^{\alpha\beta}\,\delta\gamma_{\alpha\beta} + M^{\alpha\beta}\,\delta\rho_{\alpha\beta}$. Since we assume that the directions of the external loads remain constant during deformation, there exist potential functions $\Phi(\mathbf{u}) = -\,\mathbf{P}\cdot\mathbf{u}$ and $\Psi(\mathbf{u}, \beta_\nu) = -(\mathbf{F}\cdot\mathbf{u} + \mathbf{K}\cdot\mathbf{\Omega}_t)$, such that their variations constitute the external virtual work. In this case the principle of virtual work can be transformed into a variational principle of the form $\delta I = 0$, where the functional I is given by

$$I = \iint_M [\Sigma\,(\gamma_{\alpha\beta}, \rho_{\alpha\beta}) - \mathbf{P}\cdot\mathbf{u}]\,\mathrm{d}A - \int_{C_1} (\mathbf{F}\cdot\mathbf{u} + \mathbf{K}\cdot\mathbf{\Omega}_t)\,\mathrm{d}S, \tag{55}$$

where strain–displacement relations (4), geometric boundary and corner conditions (48) and the geometric relations expressed by [14]

$$\beta_t = -\frac{1}{1 + 2\gamma_{tt} - c_\nu^2}\{c_\nu c_t \beta_\nu + c\,\sqrt{[(1 + 2\gamma_{tt})(1 - \beta_\nu^2) - c_\nu^2]}\}, \tag{56a}$$

$$\beta = -1 - \frac{1}{1 + 2\gamma_{tt} - c_\nu^2}\{c_\nu c\,\beta_\nu - c_t\,\sqrt{[(1 + 2\gamma_{tt})(1 - \beta_\nu^2) - c_\nu^2]}\}, \tag{56b}$$

have to be imposed as subsidiary conditions. On the basis of the functional I, other functionals will be defined in terms of various sets of independent variables.

Let us introduce the subsidiary conditions (4), (48) and (56) into the functional I, utilizing the Lagrange multiplier method. Then we obtain the free functional

$$\begin{aligned} I_1 = &\iint_M \{\Sigma\,(\gamma_{\alpha\beta}, \rho_{\alpha\beta}) - \mathbf{P}\cdot\mathbf{u} - N^{\alpha\beta}[\gamma_{\alpha\beta} - \gamma_{\alpha\beta}(\mathbf{u})] - M^{\alpha\beta}[\rho_{\alpha\beta} - \rho_{\alpha\beta}(\mathbf{u})]\}\,\mathrm{d}A \\ &- \int_{C_1} \{\mathbf{F}\cdot\mathbf{u} + \mathbf{K}\cdot\mathbf{\Omega}_t(\mathbf{u}, \beta_\nu, \beta_t, \beta) - \lambda_t[\beta_t - \beta_t(\mathbf{u}, \beta_\nu)] - \lambda[\beta - \beta(\mathbf{u}, \beta_\nu)]\}\,\mathrm{d}S \\ &- \int_{C_2} [\mathbf{P}\cdot(\mathbf{u} - \mathbf{u}^*) + M\,(\beta_\nu - \beta_\nu^*)]\,\mathrm{d}S - \sum_i \mathbf{F}_i\cdot(\mathbf{u}_i - \mathbf{u}_i^*). \end{aligned} \tag{57}$$

In the functional I_1, the independent functions subject to variation are three displacements $\mathbf{u}$ in M, six displacement parameters $\mathbf{u}$, β_ν, β_t and β on C, three displacement $\mathbf{u}_k$ at each corner $M_k \in C$, six strain components $\gamma_{\alpha\beta}$ and $\rho_{\alpha\beta}$ in M, six Lagrange multipliers $N^{\alpha\beta}$ and $M^{\alpha\beta}$ in M, two Lagrange multipliers λ_t and λ on C_1, four Lagrange multipliers $\mathbf{P}$ and M on C_2 and three Lagrange multipliers $\mathbf{F}_i$ at each corner $M_i \in C_2$.

Taking the first variation of I_1 and transforming the results, we have

$$\begin{aligned}\delta I_1 = \iint_M \bigg\{ & \left(\frac{\partial \Sigma}{\partial \gamma_{\alpha\beta}} - N^{\alpha\beta}\right)\delta\gamma_{\alpha\beta} + \left(\frac{\partial \Sigma}{\partial \rho_{\alpha\beta}} - M^{\alpha\beta}\right)\delta\rho_{\alpha\beta} - [\gamma_{\alpha\beta} - \gamma_{\alpha\beta}(\mathbf{u})]\delta N^{\alpha\beta} \\ & - [\rho_{\alpha\beta} - \rho_{\alpha\beta}(\mathbf{u})]\delta M^{\alpha\beta} - (\mathbf{T}^\beta|_\beta + \mathbf{P})\cdot\delta\mathbf{u}\bigg\}\, \mathrm{d}A \\ & + \int_{C_1}\bigg\{(\mathbf{T} - \mathbf{T}^*)\cdot\delta\mathbf{u} + \delta\lambda_t[\beta_t - \beta_t(\mathbf{u}, \beta_\nu)] + \delta\lambda[\beta - \beta(\mathbf{u}, \beta_\nu)] \\ & + (M_{\nu\nu} - M^*_{\nu\nu})\delta\beta_\nu - \left(\frac{1}{2}k_t - \frac{1}{2}\frac{c_\nu}{\bar{a}_t}k_\nu - \lambda_t\right)\delta\beta_t - \left(-\frac{1}{2}\frac{c_\nu}{\bar{a}_t}k - \lambda\right)\delta\beta\bigg\}\, \mathrm{d}S \\ & + \int_{C_2}[(\mathbf{T} - \mathbf{P})\cdot\delta\mathbf{u} - \delta\mathbf{P}\cdot(\mathbf{u} - \mathbf{u}^*) + (M_{\nu\nu} - M)\delta\beta_\nu - (\beta_\nu - \beta^*_\nu)\delta M]\, \mathrm{d}S \\ & + \sum_i [(\mathbf{R}^*_i - \mathbf{F}_i)\cdot\delta\mathbf{u}_i - \delta\mathbf{F}_i\cdot(\mathbf{u}_i - \mathbf{u}^*_i)] \\ & + \sum_j (\mathbf{R}_j - \mathbf{R}^*_j)\cdot\delta\mathbf{u}_j. \end{aligned} \tag{58}$$

The physical meaning of the Lagrange multipliers are given from eqn (58).

Following the way[19], a number of other free functionals and associated Lagrangian variational principles may be generated from the functional I_1.

8. DISCUSSION

Pietraszkiewicz[13] has compared his results with some close results obtained by other authors. It has been shown that the shell equations derived in [13] are consistent with the fully Lagrangian nonlinear shell theory. Therefore it is interesting to compare the results obtained by Pietraszkiewicz[6–9, 13, 16], Pietraszkiewicz and Szwabowicz[14] and the present results.

The present equilibrium equations (46) differ slightly from those of [13]. The mixed curvature tensors b^α_β, appeared in the modified tensors of change of curvature, have been approximated to the mixed curvature tensors $\bar{b}^\alpha_\beta$ in case of obtaining the equilibrium equations[13]. As a result a slight difference is found in the definition of the stress resultant components $T^{\alpha\beta}$. If we utilize the tensor of change of curvature $\kappa_{\alpha\beta}$ in place of $\rho_{\alpha\beta}$, the resulting equilibrium equations agree with those of [6–9, 13, 16, 18]. On the other hand, the equilibrium equations of [14] differ significantly from the present ones. This discrepancy may be caused by the difference of tensors of change of curvature used. In [14], a new tensor of change of curvature, defined by

$$\chi_{\alpha\beta} = -\left(\sqrt{\left(\frac{\bar{a}}{a}\right)}\,\bar{b}_{\alpha\beta} - b_{\alpha\beta}\right) + b_{\alpha\beta}\gamma^\kappa_\kappa, \tag{59}$$

has been used. This tensor is a third-degree polynomial in displacements and their derivatives. The usage of such a new tensor indicates that the small-strain assumptions are introduced at the stage of the strain–displacement relationships. As discussed in [14], the small-strain assumptions should not have been introduced at too early a stage, in order to derive the consistent shell equations. It is preferable, if possible, to utilize the irrational tensor of change of curvature where no approximation is made.

The appropriate geometric boundary conditions have not been derived in [13], since the additional terms in the virtual work on the shell boundary could not have been eliminated through integration by parts. As a geometric boundary condition for rotation[6–9], the parameter $\bar{\beta}_\nu$ defined by $\bar{\beta}_\nu = \boldsymbol{\beta}\cdot\bar{\mathbf{a}}_\nu$, has been prescribed. This parameter is defined with respect to the deformed boundary, and has been used also in the Eulerian shell equations[6–9]. Such a form of the boundary conditions is incompatible with other fully Lagrangian shell equations. The present geometric boundary conditions (48) consist of the three displacement components $\mathbf{u}$ and the parameter β_ν, describing the finite rotations at the shell boundary. These expressions agree completely with those of [14, 16, 18]. Note that the fourth parameter defined by

$$\beta_\nu = \sqrt{\left(\frac{a}{\bar{a}}\right)}\,\epsilon^{\alpha\beta}\,\epsilon_{\lambda\mu}\,\nu^\mu\,\phi_\alpha\,l^\lambda{}_{\cdot\beta} \tag{60}$$

is nonlinear with respect to displacements and their derivatives.

As for the static boundary conditions, the appropriate equations have been first derived in this paper. The effects pertaining to the term $\bar{\mathbf{n}}\cdot\delta\mathbf{u}_{,\nu}$ have not been included appropriately in the static boundary conditions[6–9, 13]. In [14], as well as [16], instead of the total finite rotation vector at the boundary, the vector $\boldsymbol{\beta}$ has been used in evaluating the external virtual work for couple. Therefore the present static boundary conditions do not agree with those of [14] and [16]. In case of using the tensor $\kappa_{\alpha\beta}$ instead of $\rho_{\alpha\beta}$, the resulting static boundary conditions agree with those of [18].

The functional I_1 obtained in this paper may be called the functional for the Hu–Washizu variational principle. In case of the geometrically nonlinear shell theory, a number of functionals have been derived[19–22]. On the basis of the nonlinear shell theory derived by Pietraszkiewicz[6], a set of 16 basic free functionals without subsidiary conditions has been constructed[20]. The Hu–Washizu variational principle for the first-approximation theory of shells has been derived[21] on the basis of the results of [14]. The present functional I_1 in eqn (57) is different from the existing ones. In the existing functionals[19–22], neither the irrational tensor of change of curvature nor the total finite rotation vector has been employed. Consequently the accurate equilibrium equations, the associated boundary conditions and other shell equations are not derived from the existing functionals. While the present functional I_1 yields the accurate basic shell equations, compatible with the Lagrangian nonlinear theory of shells undergoing finite rotations.

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